Mathematics A

YAO Jiayuan

Research Fellow in *Geophysics*

Homepage www.ntu.edu.sg/home/jiayuanyao

Email jiayuanyao@ntu.edu.sg

Office SPMS-MAS-04-07

Last updated: 2019/11/14

Chapter 1: Differentiation and Integration

1. Functions and Limits
   - Definitions and Examples
   - Limits and methods
2. Differentiation
   - Derivative, Chain Rule, Implicit Differentiation
   - Applications: Rate of Change, Maximum/Minimum, L'Hospital's Rule
3. Integration
   - Indefinite and Definite Integrals: Basic definitions, Area
   - Techniques of Integration: Substitution, by-parts, partial fraction

Chapter 2: Ordinary Differential Equations

1. Introduction
2. First Order ODE
   - Method 1: Separating variables
   - Method 2: Substituting $y = vx$
   - Method 3: Integrating Factor
   - Method 4: Bernoulli's Equation
3. Interlude: Complex Numbers
4. Second order ODE
   - Method 5: Second order homogeneous
   - Method 6: Second order inhomogeneous

Chapter 3: Series

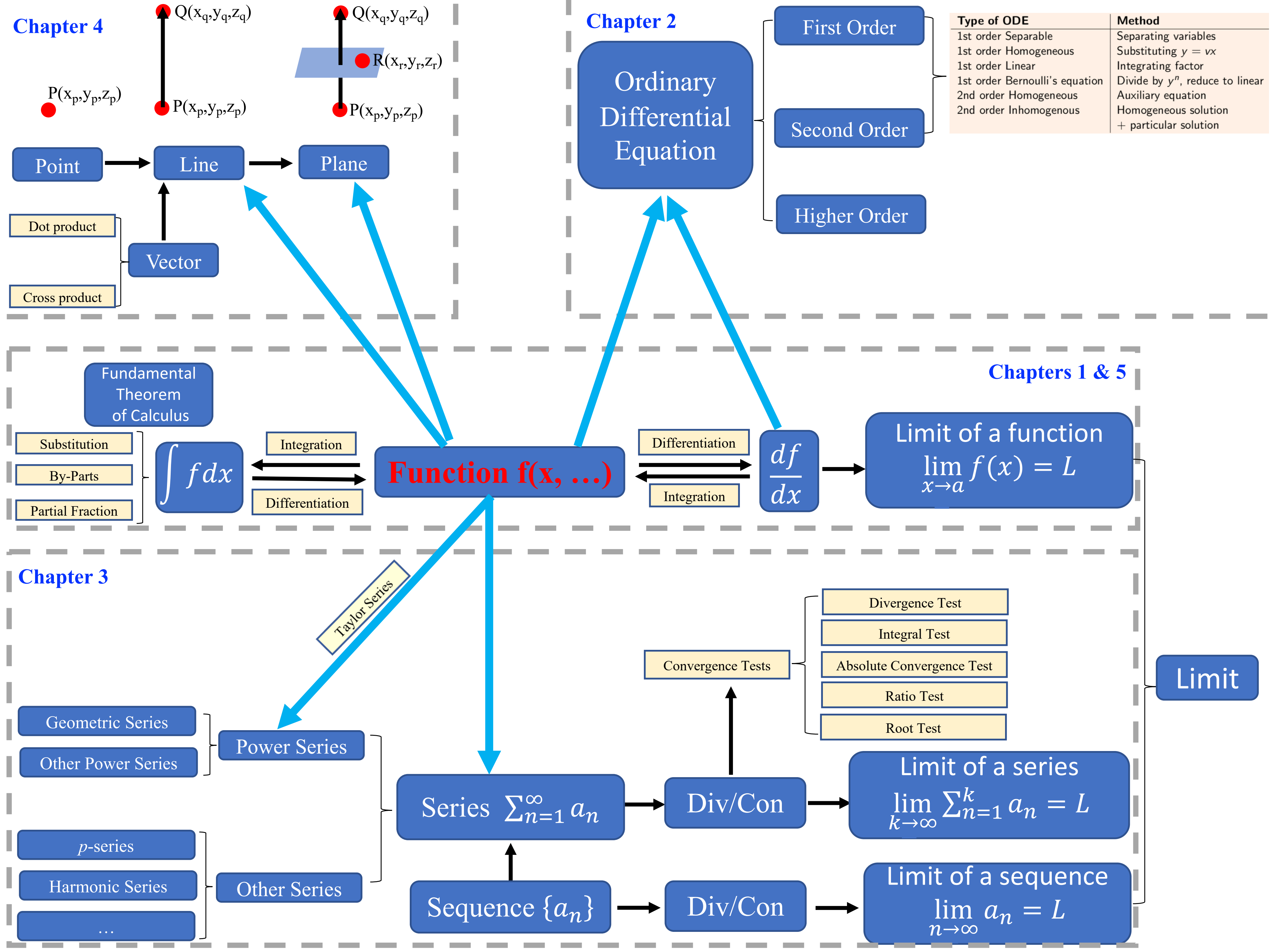
1. Sequences and Series
2. Convergence Tests for Series
   - Divergence Test
   - Integral Test
   - Absolute Convergence Test
   - Ratio Test & Root Test
3. Power Series
   - Radius of Convergence
   - Manipulating geometric series, term-by-term differentiation and integration
4. Taylor series

Chapter 4: Vectors

1. Vectors
   - Basic Properties
   - Dot Product, Projections
   - Cross Product
2. Lines
3. Planes

Chapter 5: Partial Derivatives and Multiple Integrals

1. Functions of Two Variables
2. Partial derivatives
   - Chain Rule
   - Implicit differentiation
   - Directional derivatives & Gradient vectors
3. Double Integrals
   - Meaning of double integral
   - Iterated Integral
   - Polar regions (not tested in final exam)


Chapter 4
Q(xq,yq,zq)
Q(xq,yq,zq)
R(xr,yr,zr)
P(xp,yp,zp)
P(xp,yp,zp)
P(xp,yp,zp)
Point
Line
Plane
Dot product
Vector
Cross product
Chapter 2
Ordinary Differential Equation
First Order
Second Order
Higher Order
Type of ODE | Method
1st order Separable | Separating variables
1st order Homogeneous | Substituting y = vx
1st order Linear | Integrating factor
1st order Bernoulli's equation | Divide by y^n, reduce to linear
2nd order Homogeneous | Auxiliary equation
2nd order Inhomogenous | Homogeneous solution + particular solution
Chapters 1 & 5
Fundamental Theorem of Calculus
Substitution
By-Parts
Partial Fraction
∫ f dx
Integration
Differentiation
Function f(x, …)
Differentiation
Integration
df/dx
Limit of a function
lim_{x→a} f(x) = L
Chapter 3
Taylor Series
Divergence Test
Integral Test
Convergence Tests
Absolute Convergence Test
Ratio Test
Root Test
Limit
Geometric Series
Other Power Series
Power Series
Series ∑_{n=1}^∞ a_n
Div/Con
Limit of a series
lim_{k→∞} ∑_{n=1}^k a_n = L
p-series
Harmonic Series
…
Other Series
Sequence {a_n}
Div/Con
Limit of a sequence
lim_{n→∞} a_n = L

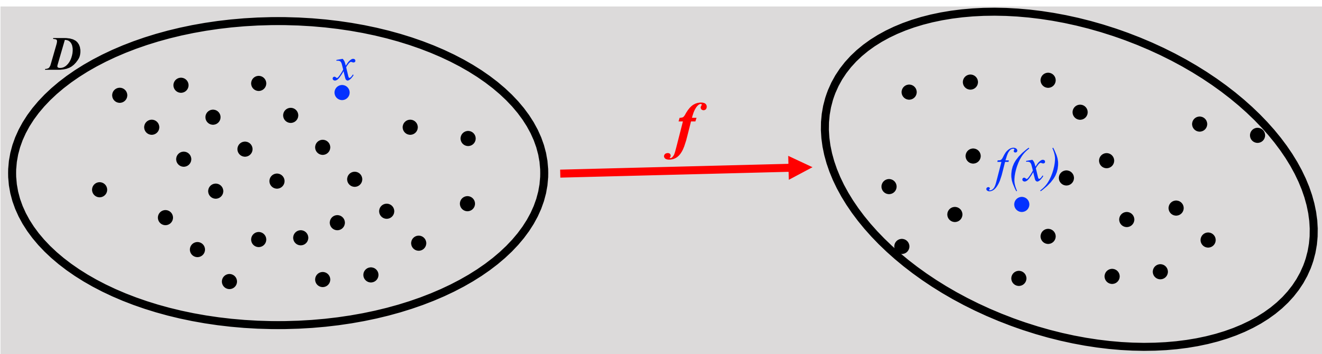
Single variable function *f(x)*

- Definition
  - It is a rule that assigns to each element *x* in a set *D* a unique element.
- Composite function:

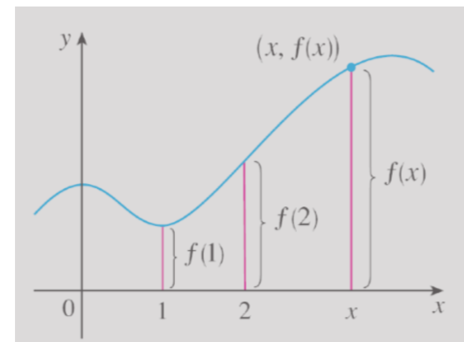
$$(g \circ f)(x) = g(f(x))$$

- Inverse function
  - Reverse process done by *f* : $(g \circ f)(x) = x$

Illustration of *f(x)*



Graph



Limit of a function

- Definition
  - The limit of *f(x)* at *a* is *L* if the value of *f(x)* approaches the real number L as *x* approaches as close as possible (but NEVER equal) to *a*.

$$\lim_{x \to a} f(x) = L$$

Derivative

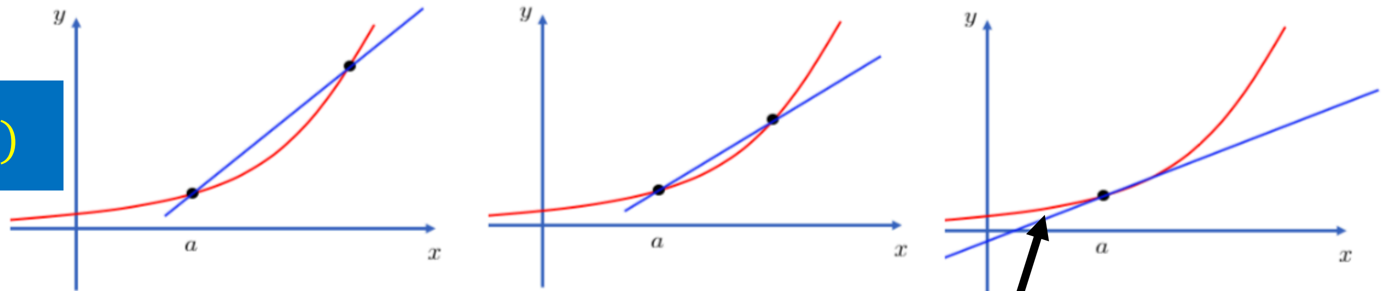
$f'(x)$ or $\frac{dy}{dx}$ or $\frac{d}{dx}f(x)$

Let $f(x)$ be a function and a be a real number. The number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the above limit exists, is called the **derivative** of $f(x)$ at $x = a$ or the slope of the tangent line of $y = f(x)$ at $x = a$.

Illustration of $f'(a)$



Tangent line at $x = a$:

$$\boldsymbol{y = f'(a)(x - a) + f(a)}$$

Four rules

Scalar coefficient: $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, k is a scalar.

Sum rule: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Product rule: $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

Chain rules

If $u = g(x)$, $y = f(u) = f(g(x))$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Applications of Derivative

Problem of finding rate of change::

- Given one rate of change $\frac{dy}{dt}$, we want to find another rate of change $\frac{dz}{dt}$
- The procedure is to find an equation that relates the two quantities y and z and then use the Chain Rule to differentiate both sides with respect to t.

The Closed Interval Method: To find the maximum and minimum values of a continuous function $f(x)$ on a closed interval $a \leq x \leq b$.

(1) Find the values of f at stationary point(s).

(2) Find the values of f at the endpoints of the interval: $f(a)$, $f(b)$.

(3) The largest of the values from Step (1) and (2) is the maximum; the smallest of the values from Step (1) and (2) is the minimum.

L'Hospital's Rule:
Suppose f and g are differentiable, and by direct substitution we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}.$$

Here a can be a real number or $\pm\infty$.

Indefinite and Definite Integrals

A function $F(x)$ is an **antiderivative** of $f(x)$ on an interval (a, b) if

$$F'(x) = f(x) \quad \text{for all } x \in (a, b).$$

Indefinite Integral:

All antiderivatives of f differ by a constant. Thus, the most general antiderivative of f on (a, b) is called the **indefinite integral** of f, and is denoted by

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$ and C is an arbitrary constant.

Definite Integral:

We obtain the **definite integral** of f over the interval $[a, b]$, denoted by $\int_a^b f(x)\,dx$, by subtracting the value of an antiderivative $F(x)$ at a from that at b:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

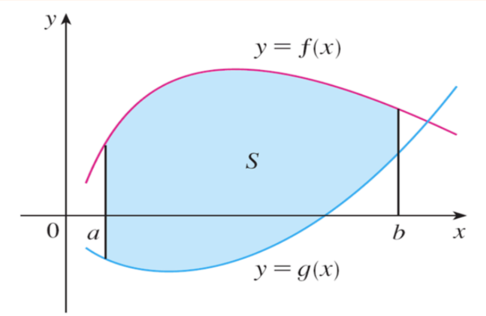
where $F(x)$ is an antiderivative of $f(x)$.

Fundamental Theorem of Calculus.

Area between two curves:

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b (\underbrace{f(x)}_{\text{top curve}} - \underbrace{g(x)}_{\text{bottom curve}})\,dx.$$



$$A = \int_a^b (\underbrace{f(x)}_{\text{top curve}} - \underbrace{g(x)}_{\text{bottom curve}})\,dx.$$

Techniques of Integration

Substitution Rule:
Steps when applying the Substitution Rule to integrate

$$\int_a^b f(x)\,dx$$

- Think of a function $u = g(x)$.
- Compute $\frac{du}{dx} = g'(x) \implies dx = \frac{1}{g'(x)}\,du$.
- Convert $f(x)\,dx$ into an expression in terms of u and du.
- Replace the lower limit a by $g(a)$, and the upper limit b by $g(b)$.
- Integrate with respect to u.

Integration by Parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

In the case of a definite integral, we have

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Partial Fraction Decomposition:

- Start with a rational function $\frac{P(x)}{Q(x)}$, where the degree of P is **strictly less** than the degree of Q.

  This is important. Otherwise, we will first do a **long division** to expand the function.

- We factor the denominator $Q(x)$ as completely as possible into irreducible factors. For our purposes, $Q(x)$ only contains **linear** or **quadratic** factors.

Each factor in the denominator (and their multiplicity $r = 1, 2, 3, \cdots$) will determine the term(s) that occur in the partial fraction decomposition.

Factor in $Q(x)$	**Term in partial fraction decomposition**
$(ax+b)^r$	$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$
$(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Ordinary Differential Equation

Type of ODE	Method	
1st order Separable	Separating variables	Method 1
1st order Homogeneous	Substituting $y = vx$	Method 2
1st order Linear	Integrating factor	Method 3
1st order Bernoulli's equation	Divide by y^n, reduce to linear	Method 4
2nd order Homogeneous	Auxiliary equation	
2nd order Inhomogenous	Homogeneous solution + particular solution	

Method 1: Separating variables

- We can 'separate' the y-factors and the x-factors into opposite sites:

$$g(y)\,dy = f(x)\,dx.$$

- Then solve the ODE by integrating both sides:

$$\int g(y)\,dy = \int f(x)\,dx.$$

$$\frac{dy}{dx} + Py = Q$$

Method 3: Integrating Factor

- Multiply both sides of the equation by $e^{\int P\,dx}$, called the **integrating factor**. (When evaluating $\int P\,dx$, we will ignore any constant.)
- This converts the left-hand side of the ODE into the derivative of the product $ye^{\int P\,dx}$.
- Integrate both sides to obtain a solution.

Method 2: Substituting y=vx (For 1st Order Homogeneous ODE)

- Let $y = vx$.
- Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ by Product Rule.
- Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation. This converts 1st order homogeneous ODE into a separable ODE in terms of v and x for which we can solve by separating variables.
- Convert back into terms of y and x.

Method 4: Bernoulli's Equation

- To solve $\frac{dy}{dx} + Py = Qy^n$, first divide both sides by y^n.
- Put $z = y^{1-n}$ and convert the ODE into a linear 1st order ODE:

$$\frac{dz}{dx} + P^*z = Q^*,$$

where $P^* = (1-n)P$, $Q^* = (1-n)Q$.
- Solve the linear ODE using the method of integrating factor.
- Convert back into terms of y and x.

Ordinary Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

Type of ODE	Method	
1st order Separable	Separating variables	
1st order Homogeneous	Substituting $y = vx$	
1st order Linear	Integrating factor	
1st order Bernoulli's equation	Divide by y^n, reduce to linear	
2nd order Homogeneous	Auxiliary equation	Method 5
2nd order Inhomogenous	Homogeneous solution + particular solution	Method 6

Method 5: Homogeneous second order ODE

- Let w be a variable, and m_1, m_2 be the roots of the auxiliary equation $aw^2 + bw + c = 0$.
- Recall that the roots of the auxiliary (quadratic) equation is given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Let m_1, m_2 be the roots of the auxiliary equation. There are three cases:
  (i) Real and distinct roots i.e. $m_1 \neq m_2$. The general solution is

$$y = Ae^{m_1 x} + Be^{m_2 x}.$$

  (ii) Real and equal roots i.e $m_1 = m_2 = m$. The general solution is

$$y = (Ax + B)e^{mx}.$$

  (iii) Complex roots i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. The general solution is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Here, A and B are arbitrary constants.

Method 6: Inhomogeneous second order ODE

- Find the general solution $y_h(x)$ of the homogeneous ODE $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. It is called the homogeneous solution.
- Find ANY solution $y_p(x)$ of the non-homogeneous ODE $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. It is called a particular solution.
- The general solution to (2) is

$$y = y_h(x) + y_p(x).$$

- First, we guess the form of $y_p(x)$ based on the form of $r(x)$. This is given in the table below.

$r(x)$	$y_p(x)$
Kx^n	$C_n x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$
$Ke^{\alpha x}$	$Ce^{\alpha x}$
$K\cos(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$K\sin(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$Ke^{\alpha x}\cos(\beta x)$	$e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$
$Ke^{\alpha x}\sin(\beta x)$	$e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$

- The various constants appearing in $y_p(x)$ can be determined by assuming that $y_p(x)$ satisfies the ODE.

Sequences

A **sequence** can be regarded as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots.$$

Limit of a sequence:

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that terms of the sequence $\{a_n\}$ approach L as n gets larger and larger.

- If L is a real number, then we say that $\{a_n\}$ **converges** to L (or is **convergent**).
- Otherwise, we say that the sequence **diverges** (is **divergent**).

Series

If we add the terms of a sequence $\{a_n\}_{n=1}^{\infty}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

which is called a **series**

Limit of Series $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} a_n}_{\text{partial sum}}$.

If the above limit exists and is equal to S, then the series $\sum a_n$ is called **convergent**, and the number S is called the **sum** of the series. Otherwise, the series is said to be **divergent**.

Geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

- converges to $\dfrac{a}{1-r}$, if $|r| < 1$.
- diverges is $|r| \geq 1$.

Harmonic series (divergent)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

- convergent if $p > 1$;
- divergent if $p \leq 1$.

Convergence Tests for Series

(i) Divergence Test
(ii) Integral Test
(iii) Absolute Convergence Test
(iv) Ratio Test
(v) Root Test

Convergence Tests for Series

Divergence Test:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$,

then the series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

The Integral Test:
Suppose f is a **continuous**, **positive**, **decreasing** function on $[c, \infty)$, and let $a_n = f(n)$.

(i) If $\int_c^{\infty} f(x)\,dx$ is **convergent (i.e. equals a real number)**, then the series $\sum_{n=c}^{\infty} a_n$ is **convergent**.

(ii) If $\int_c^{\infty} f(x)\,dx$ is **divergent**, then $\sum_{n=c}^{\infty} a_n$ is **divergent**.

Absolute Convergence Test:

If $\sum |a_n|$ converges, then the series $\sum a_n$ is convergent.

Ratio Test:

Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

(i) If $\rho < 1$, then $\sum a_n$ **converges absolutely** (so it converges by the Absolute Convergence Test).
(ii) If $\rho > 1$ or $\rho = \infty$, then $\sum a_n$ **diverges**.
(iii) If $\rho = 1$, then Ratio Test is inconclusive (the series may converge or diverge).

Root Test:

Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(i) If $L < 1$, then $\sum a_n$ **converges absolutely**.
(ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ **diverges**.
(iii) If $L = 1$, the Root Test is inconclusive. The series may converge or diverge.

Power series

A **power series centred** at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is a variable, and the c_n's are constants called the **coefficients** of the series.

Manipulating geometric series

The goal is to use the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \qquad (1)$$

which we know is convergent for all $|x| < 1$, to express a given function as a power series.

Term-by-term differentiation and integration:
Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

has radius of convergence $R > 0$. Then

(i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Moreover, the series in (i) and (ii) have the **same radius of convergence** R.

Taylor Series:

If f has a power series expansion (representation) at $x = a$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Here, $f^n(a)$ is the n-th derivative of f at $x = a$.

- The power series on the right-hand side is called the **Taylor series of $f(x)$ centred at** $x = a$.
- In the special case $a = 0$, the Taylor series is also called the **Maclaurin series**.

Vector

A vector is completely defined by two things:

- Length
- Direction

Normalization:

If $\mathbf{a} \neq \mathbf{0}$, then the unit vector in the same direction as $\mathbf{a}$ is given by

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \underbrace{\frac{1}{\|\mathbf{a}\|}}_{\text{positive scalar}} \cdot \mathbf{a}$$

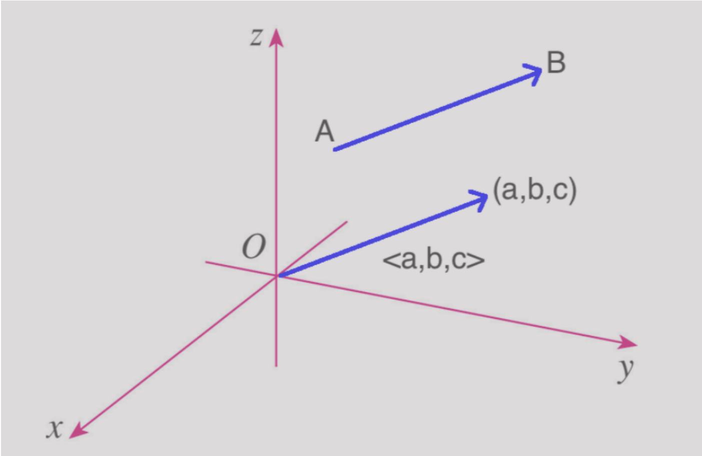
Dot product

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

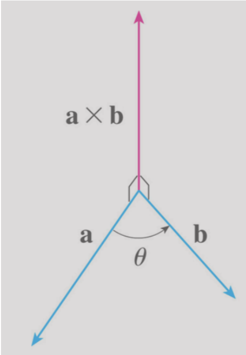
Dot product angle formula:

Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta.$$



Cross product



$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} \\
&\quad + (a_1 b_2 - a_2 b_1)\mathbf{k}.
\end{aligned}$$

Cross product angle formula:

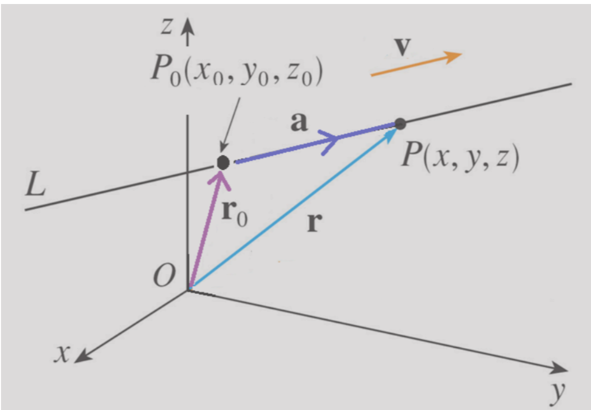
If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$

Line

To locate a line L in space, we need

- A point say $P_0(x_0, y_0, z_0)$ on the line L.
- A vector $\mathbf{v}$ whose direction is parallel to the line L.



Vector equation of a line

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$$
$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

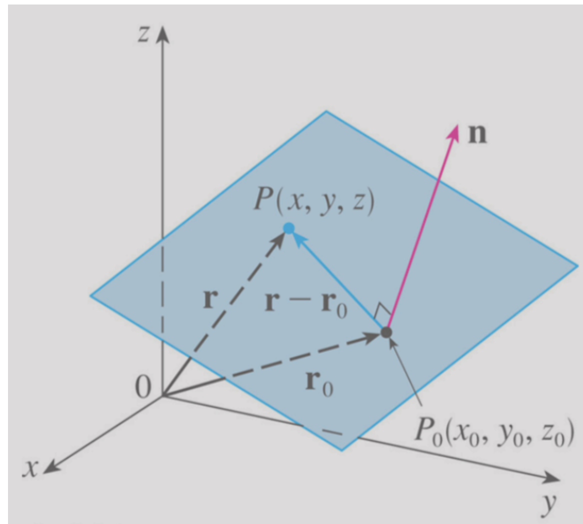
Parametric Equation of Line:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Plane

To locate a particular plane in space, we need

- A point say $P_0(x_0, y_0, z_0)$ on the plane.
- A vector $\mathbf{n}$ whose direction is perpendicular to the plane.



Vector Equation of Plane:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be written as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$$

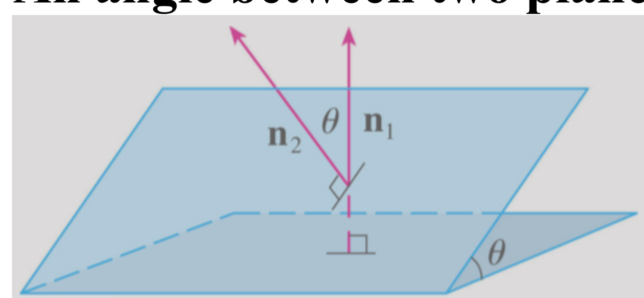
Linear Equation of Plane:

$$ax + by + cz = d,$$

where

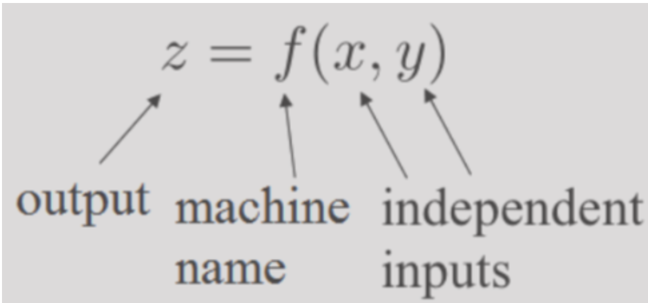
$$d = ax_0 + by_0 + cz_0.$$

An angle between two planes

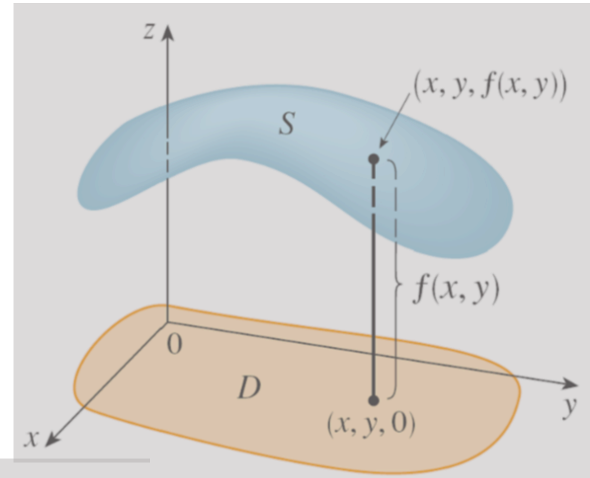


Functions of Two Variables

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ a unique real number denoted by $f(x, y)$.



Graph



Partial Derivatives

(i) the **partial derivative of** $f(x, y)$ **with respect to** x by treating y as a constant and simply differentiating $f(x, y)$ with respect to x,

(ii) the **partial derivative of** $f(x, y)$ **with respect to** y by treating x as a constant and simply differentiating $f(x, y)$ with respect to y.

Second order partial derivatives

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

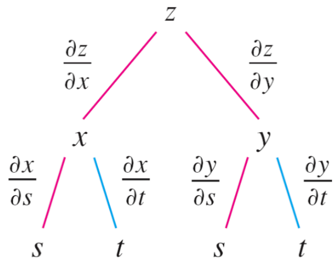
$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem:

Suppose f is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial Derivatives of multivariable function



The Chain Rule I

Suppose that $z = f(x, y)$ is a function of x and y, and $x = g(s, t)$ and $y = h(s, t)$ are both functions of s and t. Then,

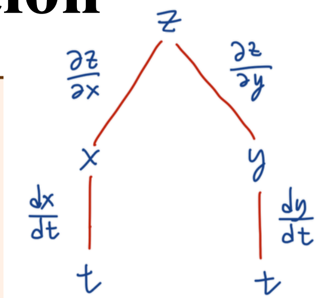
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

The Chain Rule II

Suppose that $z = f(x, y)$ is a function of x and y, and $x = g(t)$ and $y = h(t)$ are both (single-variable) functions of t. Then,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$



Implicit Differentiation:

Suppose the equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

provided $F_z(x, y, z) \neq 0$.

Directional Derivative:

Given a function $f(x, y)$ and a **unit direction vector u** $= \langle u_1, u_2 \rangle = u_1\mathbf{i} + u_2\mathbf{j}$. The rate of change of f along $\mathbf{u}$ at the point (a, b), called the **directional derivative** of f **along u**, is given by

$$D_{\mathbf{u}} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2.$$

Gradient vector

The **gradient** (or **gradient vector**) of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = \langle f_x, f_y \rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Gradient vector is normal to level curve:
Fix (x_0, y_0). Let $f(x, y) = k$ be the level curve such that $f(x_0, y_0) = k$. Then $\nabla f(x_0, y_0)$ is **perpendicular/normal** to the level curve $f(x, y) = k$.

Gradient vector is normal to level surface:
Fix (x_0, y_0, z_0). Let $F(x, y, z) = k$ be the level surface such that $F(x_0, y_0, z_0) = k$. Then $\nabla F(x_0, y_0, z_0)$ is **perpendicular/normal** to the level surface $F(x, y, z) = k$.

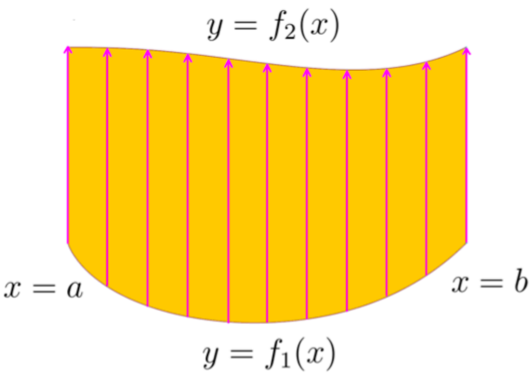
Maximizing rate of change:

The maximum value of $D_{\mathbf{u}} f(a, b)$ is $\|\nabla f(a, b)\|$, and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(a, b)$.

The minimum value of $D_{\mathbf{u}} f(a, b)$ is $-\|\nabla f(a, b)\|$, and it occurs when $\mathbf{u}$ has the opposite direction as $\nabla f(a, b)$.

Double Integrals

Type I



$$\iint_D f(x, y)\, dA$$

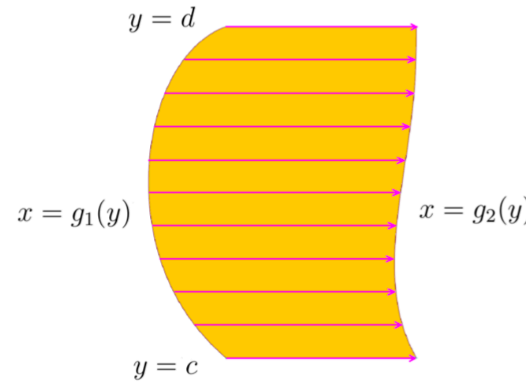
- **Type I region**:

$$D = \{(x, y) : a \leq x \leq b, \quad f_1(x) \leq y \leq f_2(x)\}.$$

- **Type II region**:

$$D = \{(x, y) : c \leq y \leq d, \quad g_1(y) \leq x \leq g_2(y)\}$$

Type II



Evaluating double integral via iterated integrals:

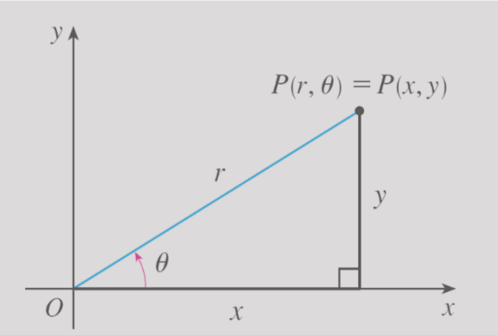
- For Type I region
  $D = \{(x, y) : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}$:

$$\iint_D f(x)\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy\, dx$$

- For Type II region
  $D = \{(x, y) : c \leq y \leq d,\ g_1(x) \leq x \leq g_2(y)\}$:

$$\iint_D f(x)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy$$

Polar Coordinates



Change to Polar Coordinates in Double Integral:

If f is continuous on a polar rectangle R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Relationship between (r, θ) and (x, y)

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin$$